Engineering Notes

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Frequencies of the Alternating Forces due to Interactions of Contrarotating Propellers

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Introduction

SOMEWHAT to our surprise, we have not found any published discussion of the frequencies of the fluctuating forces developed by interactions between a pair of contrarotating propellers. Although the calculation of the expected frequencies of the alternating thrust and torque is straightforward, the calculation for side forces involves sufficient subtlety to warrant documentation.

This Note is concerned with the frequencies of the forces only, and not with their magnitude. Nevertheless, it becomes apparent during the determination of frequencies that the fluctuating interaction forces are zero for certain combinations of the blade numbers of the two propellers.

Symbols and Statement of the Problem

Suppose the forward propeller has Y blades,and the aft propeller Z blades. The individual blades of the forward propeller are labeled with an index y going from zero to $(Y-1)$, and those of the aft propeller with an index z going from zero to $(Z-1)$. (The Yth and Zth blades are identical to the zeroth blades.) The angles between the upward vertical and a reference radial line on the forward yth blade is called θ_y, and the equivalent angle for the zth aft blade is called ϕ_z.

If the forward propeller rotates counterclockwise at M revolutions per unit time, and the aft propeller rotates clockwise at N revolutions per unit time, then the angles will vary with time as

$$\theta_y = -2\pi Mt + 2\pi(y/Y) \tag{1}$$

and

$$\phi_z = 2\pi Nt + 2\pi(z/Z) + \varphi_0 \tag{2}$$

The fixed angle φ_0 in the aforementioned indicates that the aft zeroth blade makes an angle φ_0 to the vertical at time $t=0$. The omission of an equivalent fixed angle for the forward propeller indicates that the time origin is chosen so $t=0$ when $\theta_0 = 0$.

Fluctuating forces will be developed by the aft propeller in response to nonuniformities in the velocity field induced by the forward propeller, and forces will be developed by the forward propeller because of nonuniform velocities induced by the aft propeller. These are the only forces that will be developed if the flow into the forward propeller is axisymmetric. If the flow into the forward propeller is not axisymmetric, additional fluctuating forces will be generated by each propeller in response to these preexisting nonuniformities. But the frequencies of the latter forces are already well known, since they are identical to those developed by a single propeller in a nonuniform inflow. This is so because the various nonuniform velocities contribute to the fluctuating forces by superposition (or at least it is so assumed in the conventional first-order linear theory). Accordingly, this Note will be concerned only with the forces resulting from interaction between the two propellers.

Detailed Analysis

The fluctuating side force developed by the aft propeller will be considered first. The fluctuating tangential force T_z developed by the zth aft blade is assumed to be a periodic function of the angle Ω_z between that blade and the zeroth forward blade. Since all aft blades are identical, the form of this function will be the same for each aft blade. Since all Y forward blades are identical with each other, the velocity field induced by each forward blade will be identical to the others. Thus, the velocity field and associated fluctuating forces induced by the forward propeller will have an angular periodicity of $(2\pi/Y)$ rather than simply 2π.

Accordingly, the periodic tangential force on the zth aft blade can be expressed as a Fourier series

$$T_z = \sum_{k=1}^{\infty} B_k \cos(kY\Omega_z - \psi_k) \tag{3}$$

where Ω_z, the angle between the zth aft blade and the zeroth forward blade, is given by

$$\Omega_z = \phi_z - \theta_0 \tag{4}$$

The horizontal component of this tangential force is given by

$$H_z = T_z \cos\phi_z \tag{5}$$

The total horizontal force is obtained by summing over the Z blades, viz.

$$H = \sum_{k=1}^{\infty} \sum_{z=0}^{Z-1} B_k \cos[kY(\phi_z - \theta_0) - \psi_k] \tag{6}$$

Using the identity $\cos a \cos b = ½\cos(a+b) + ½\cos(a-b)$, Eq. (6) becomes

$$H = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{z=0}^{Z-1} B_k \{\cos[(kY+1)\phi_z - kY\theta_0 - \psi_k] + \cos[(kY-1)\phi_z - kY\theta_0 - \psi_k]\} \tag{7}$$

After introducing the time dependence of the blade angles ϕ_z and θ_0 from Eqs. (1) and (2) into Eq. (7), the latter may be written in symbolic form as

$$H = \frac{1}{2} \sum_{k=1}^{\infty} B_k [\Sigma_+ + \Sigma_-] \tag{8a}$$

Received Dec. 8, 1975.
Index category: Marine Propulsion System Integration.
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where

$$\Sigma_{\pm} \equiv \sum_{z=0}^{Z-1} \cos[a_{\pm} + 2\pi(z/Z)(kY \pm 1)] \tag{8b}$$

with

$$a_{\pm} = (kY \pm 1)(2\pi Nt + \varphi_0) + 2\pi kYMt - \psi_k \tag{8c}$$

When written in this form the entire z-dependence of each term of the sum appears in one place in Eq. (8b).

The sums over z are of a standard form encountered in the theory of fluctuating propeller forces. Their values are given by the form

$$\sum_{r=0}^{R} \cos(x+ry) = \cos(x + \frac{1}{2} Ry) \frac{\sin\frac{1}{2}(R+1)y}{\sin\frac{1}{2}y}$$

(See, e.g., formula 3.61-9 in "Smithsonian Mathematical Formulae," Publication 2672 of the Smithsonian Institution, 1939.) In the present case, $x = a_{\pm}$, $y = (2\pi/Z)(kY \pm 1)$, and $R = (Z-1)$. Examination of the quotient of the two sines,

$$\frac{\sin\pi(kY \pm 1)}{\sin(\pi/Z)(kY \pm 1)}$$

indicates that the numerator is always zero, since k and Y are integers. Accordingly, the entire sum will be zero except if the denominator is also zero, which is the case when

$$kY \pm 1 = mZ \tag{9}$$

where m is an integer. When this occurs, the quotient has the indeterminate form (0/0) which can be evaluated to

$$\frac{\sin\pi mZ}{\sin\pi m} = Z(-1)^{m(Z-1)} \tag{10}$$

Equation (9) indicates that side forces will be generated by interaction between the propellers only for certain values of k and m, depending on the blades number Y and Z of the two propellers. If Eq. (9) is satisfied with the plus sign, the sum Σ_+ will be nonzero while Σ_- will be zero, and vice versa. In either case, the sum has the form

$$\Sigma_+ + \Sigma_- = Z(-1)^{m(Z-1)} \cos[a + \pi m(Z-1)] = Z\cos a \tag{11}$$

where a is now used instead of a_+ or a_-, since they have the same value

$$a = mZ(2\pi Nt + \varphi_0) + 2\pi kYMt - \psi_k \tag{12}$$

Final Results for Side-Force Frequencies

The total horizontal force can now be written by inserting Eqs. (11) and (12) into Eq. (8a). The final result is

$$H = \frac{1}{2} Z \sum_k B_k \cos[2\pi(mZN + kYM)t + mZ\varphi_0 - \psi_k] \tag{13}$$

where k and m are integers whose values are restricted to those satisfying the relation

$$kY \pm 1 = mZ \tag{14}$$

The cyclic frequencies associated with these allowed values are given by

$$f_{km} = kYM + mZN \tag{15}$$

If Eq. (14) is satisfied by $k=1$ and $m=1$, then the lowest frequency will simply be the sum of the blade frequencies of the 2 propellers. This occurs whenever $Z = Y \pm 1$; *i.e.*, when the two propellers differ by one blade. For higher values of k and m, the frequencies will equal the sum of the appropriate harmonics of the two blade frequencies. Table 1 lists the allowed frequencies for various blade combinations when both propellers are contrarotating at the same speed. Note that no side forces are generated when both propellers have an even number of blades (not necessarily the same number).

This calculation has been carried out for the horizontal force. The results can be applied to the vertical force simply by defining the angle φ_0, first appearing in Eq. (2), to be measured relative to the horizontal rather than the vertical; and the same for the other angles. The frequencies and magnitudes are identical for both vertical and horizontal forces; only the phase is different, the difference being an angle $(\pi mZ/2)$.

The calculation of the frequencies of the force generated by the forward propeller is also identical except that the symbols k, Y, and M are associated with the aft propeller, and m, Z, and N with the forward propeller. The frequencies of the forces are the same for both propellers. The magnitudes of the forces generated by the two propellers will generally be different, however.

Table 1 Frequencies of alternating forces for contrarotating propellers (expressed as harmonics of shaft frequency)[a]

Number of blades of other propeller	Number of blades of one propeller: 3	4	5	6	7
				Side forces	
3	6/12	7/17	11/19	...	13/29
4	24/48	8/16	9/31	...	15/41
5	30/60	40/80	10/20	11/49	29/41
6	12/24	24/48	60/120	12/24	13/71
7	41/84	56/112	70/140	84/168	14/28
		Thrust & torque			

[a]The numbers in the table give the two lowest frequencies, expressed as harmonics of one shaft frequency, for two propellers with the indicated number of blades contrarotating at the same speed. The values above the staggered dividing line are for alternating side forces, and those below are for alternating thrust and torque. The table does not distinguish between fore and aft propellers because the frequencies are the same for both of them and do not depend on which propeller is forward.

A simple procedure for determining the frequencies for side forces as given by Eq. (15) is to write out the sequences of integer multiples of the blade numbers. The frequencies at which side forces will occur are those given by the sum of those numbers which differ by unity. For example, for a contrarotating pair of 3 and 5 blades:

3:	3	6	9	12	15	18	21...
5:		5	10		15		20...
∴ frequencies		11	19				41...

For thrust and torque, the frequencies are twice the least common multiples of the two blade numbers and higher multiples of these. Thus for a combination of 4- and 6-blades, the least common multiple is 12 and the frequencies for these forces are 24, 48, etc.

Alternating Thrust and Torque

For completeness, the allowed frequencies for alternating thrust and torque are given. The cyclic frequencies are

$$k'YM + m'ZN \tag{16}$$

but now k' and m' are integers which satisfy

$$k'Y = m'Z \tag{17}$$

When both propellers have the same number of blades, the lowest frequency equals the sum of the separate blade frequencies.

Both Propellers Rotating in Same Direction

The results may be applied to 2 propellers rotating in the same direction by simply using either $-M$ or $-N$ in the equations, as the case may be, instead of the positive value. A situation worthy of some discussion is that which occurs when both propellers rotate at the same speed in the same direction. This is the case of two propellers on one shaft.

Suppose the speed of both propellers is N in the same direction. Then the formulas apply with $M=-N$. The frequencies of the side forces are given by Eq. (15) as

$$f_{km}=N(kY-mZ) \tag{18}$$

However, Eq. (14) requires that $|kY-mZ|=1$. Accordingly, a side force will be developed at shaft frequency for all allowed values of k and m. It may seem surprising that an alternating side force can be developed when both propellers turn together. This occurs because the position of the aft propeller relative to the forward one results in differences in the steady tangential force on different blades. This again results in a net unbalanced side force of constant magnitude rotating at shaft frequency, just as would be the case for a 1-bladed propeller. There is no fluctuating thrust or torque in this case; when $M=-N$ the combination of Eqs. (16) and (17) is satisfied only at zero frequency.

Magnitude of the Forces

No attempt is made here to calculate the magnitude of the fluctuating forces, these being proportional to the unevaluated coefficients B_k. Nevertheless, it may be useful to mention certain factors which can be shown to influence the magnitude of the forces without requiring detailed analysis.

The magnitude of the force associated with a particular value of the harmonic index k depends on the value of B_k. Generally, B_k is largest for $k=1$ and decreases with increasing k. Accordingly, it is usually desirable to choose propeller combinations so that the allowed frequencies are associated only with relatively large values of k.

If there is concern with side forces only, then a pair of even bladed propellers should be used, because there are no interaction side forces at all when both propellers have an even number of blades. Such a pair will have alternating thrust and torque, however. For a combination of a 4-blader forward and 6-blader aft, for example, alternting thrust and torque will occur at a frequency equal to twice the aft blade frequency plus 3 times the forward one, e.g., at 24 times shaft frequency if both are contrarotating at the same speed.

If side forces are of major concern, the worse choice is a combination where the blade numbers differ by one. Such a combination is good, however, if only alternating thrust and torque are of concern. When side forces as well as alternating thrust and torque are of concern, the best compromise may be a 5 blader forward and 7 blader aft, or perhaps a 4 blader forward and 6 blader aft.

Finally, the reader is reminded that all of the above applies only to forces resulting from interaction between the two propellers. These will be the only forces when the inflow to the forward propeller is axisymmetric. If the inflow is not axisymmetric, additional fluctuating forces will be superposed upon those discussed in this Note.